A new bound on the capacity of the binary deletion channel with high deletion probabilities

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***Abstract*— Let $C(d)$ be the capacity of the binary deletion channel with deletion probability d. It was proved by Drinea and Mitzenmacher that, for all d, $C(d)/(1-d) \geq 0.1185$. Fertonani and Duman recently showed that $\limsup_{d\to 1} C(d)/(1-d) \leq 0.49$. In this paper, it is proved that $\lim_{d\to 1} C(d)/(1-d)$ exists and is equal to $\inf_d C(d)/(1-d)$. This result suggests the conjecture that the curve $C(d)$ my be convex in the interval $d \in [0,1]$. Furthermore, using currently known bounds for $C(d)$, it leads to the upper bound $\lim_{d\to 1} C(d)/(1-d) \leq 0.4143$.**

I. Introduction

A binary deletion channel W^d is defined as a binary channel that drops bits of the input sequence independently with probability d. Those bits that are not dropped simply pass through the channel unaltered. While simple to describe, the deletion channel proves to be very difficult to analyze. Dobrushin ([1]) showed that for such a channel it is possible to define a capacity $C(d)$ and that a Shannon like theorem applies to this channel. However, no closed formula expression is known up to now for the capacity $C(d)$, and only upper and lower bounds are currently available (see [2], [3], [4], [5], [6]).

For small values of d, it was recently independently proved in [4] and [5] that $C(d) \approx 1-H(d)$, where $H(d)$ is the binary entropy function. For values of d close to 1, it is known (see [7], [6]) that $C(d)$ satisfies

$$0.1185 \leq \liminf_{d\to 1} \frac{C(d)}{1-d} \leq \limsup_{d\to 1} \frac{C(d)}{1-d} \leq 0.49 \qquad (1)$$

As far as the author knows, there is no result in the literature on the existence of $\lim_{d\to 1} C(d)/(1-d)$. In this paper, it is proved that the limit exists and, in particular, that

$$\lim_{d\to 1} \frac{C(d)}{(1-d)} = \inf_d \frac{C(d)}{(1-d)}. \qquad (2)$$

The best currently known upper bound for $C(d)$, when used in the right hand side of (2), leads to the upper bound

$$\lim_{d\to 1} \frac{C(d)}{(1-d)} \leq 0.4143, \qquad (3)$$

which improves the best previously known bound of equation (1). Furthermore, equation (2) suggests the conjecture that $C(d)$ may be a convex function of d. Indeed, as discussed in Section IV below, experimental evidence (see Figure 1) suggests the convexity of $C(d)$ for values of d sufficiently smaller than 1, while it is not easy to exclude that the function may be concave near $d = 1$. Equation (2) is only a necessary condition[1] for the convexity of $C(d)$ near $d = 1$. It is, however, sufficient to conclude that $C(d)$ is not strictly concave in any neighborhood of $d = 1$. Thus, either $C(d)$ exhibit a pathological behavior near $d = 1$, or it is convex in a sufficiently small neighborhood of $d = 1$. A proof of the convexity of $C(d)$ would of course imply equation (2) and thus equation (3).

The main idea used in this paper is the intuitive fact that, for a large enough number of input bits n, the deletion channel W^d is fairly well approximated by a channel which drops exactly $[dn]$ bits selected uniformly at random. In particular, we show that a channel $W_{n,k}$ with n-bits input and k-bits output, selected uniformly within the k-bits subsequences of the input, has a capacity that is close to $C(1-k/n)$ for large enough n. Using this result, we build upon the work in [6] to prove (2).

II. Definition and regularity of $C(d)$

For any i and j, let $X_i^j = (X_i, X_{i+1}, \ldots, X_j)$ and, similarly $Y_i^j = (Y_i, Y_{i+1} \ldots, Y_j)$. Let W_n^d be a channel with an n-bit string input whose output is obtained by dropping the bits of the input independently with probability d. Let then

$$C_n(d) = \frac{1}{n} \max_{p_{X_1^n}} I(X_1^n; W_n^d(X_1^n)). \qquad (4)$$

It was proved by Dobrushin [1] that a transmission capacity $C(d)$ can be consistently defined for the deletion channel W^d and that it holds

$$C(d) = \lim_{n\to\infty} C_n(d). \qquad (5)$$

Figure 1 shows the graph of the $C_n(d)$ functions for $n = 1, \ldots, 17$. The main objective of this section is to study the convergence of the $C_n(d)$ functions to deduce a regularity result for $C(d)$.

The following lemma gives a quantitative bound on the rate of convergence in (5).

Lemma 1: (see also [1], [4], [6]) For every $d \in [0,1]$ and $n \geq 1$

$$C_n(d) - \frac{\log(n+1)}{n} \leq C(d) \leq C_n(d). \qquad (6)$$

[1] It is not difficult to construct examples of "pathological" functions $f(d)$ that satisfy equation (2), when used in place of $C(d)$, but are not convex in any neighborhood of $d = 1$.

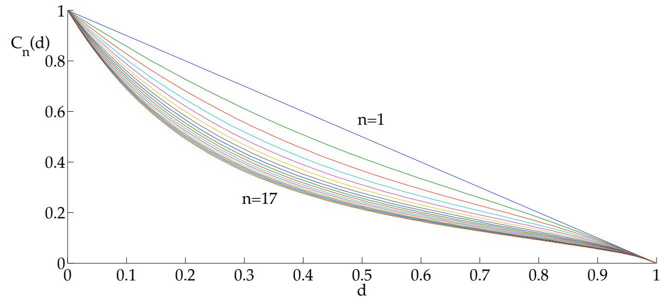

Fig. 1. Plot of the $C_n(d)$ functions for $n = 1 \dots 17$ obtained by numerical evaluations in [6].

Proof: As observed in [4], $nC_n(d)$ is a subadditive function of n. In fact, for an input X_1^{n+m}, let $\tilde{Y}_{(0)} = W_n^d(X_1^n)$ and $\tilde{Y}_{(1)} = W_m^d(X_{n+1}^{n+m})$. Note that $Y = W_{n+m}^d(X_1^{n+m})$ can be obtained as a concatenation of the strings $\tilde{Y}_{(0)}$ and $\tilde{Y}_{(1)}$. Thus, $X_1^{n+m} \to (\tilde{Y}_{(0)}, \tilde{Y}_{(1)}) \to Y$ is a Markov chain. Hence,

$$\begin{aligned}(n+m)C_{n+m}(d) &= \max_{p_{X_1^{n+m}}} I(X_1^{n+m}; Y) \\ &\leq \max_{p_{X_1^{n+m}}} I(X_1^{n+m}; (\tilde{Y}_{(0)}, \tilde{Y}_{(1)})) \\ &\leq nC_n(d) + mC_m(d).\end{aligned}$$

This implies by Fekete's lemma (see [8, Prob. 98]) that the limit $C(d) = \lim_{n\to\infty} C_n(d)$ exists and it satisfies $C(d) = \inf_{n\geq 1} C_n(d)$. This proves the right hand side inequality.

Take now an integer $h > 1$ and consider, for an input X_1^{hn}, the output $Y = W_{hn}^d(X_1^{hn})$ as the concatenation of the h outputs $\tilde{Y}_{(i)} = W_n^d(X_{ni+1}^{ni+n})$, $i = 0, \dots, h-1$. Let for convenience $\tilde{Y}_{(0)}^{(h-1)} = (\tilde{Y}_{(0)}, \tilde{Y}_{(1)}, \dots, \tilde{Y}_{(h-1)})$. It is clear that $X_1^{hn} \to \tilde{Y}_{(0)}^{(h-1)} \to Y$ is a Markov Chain. Let L_i be the length of $\tilde{Y}_{(i)}$. We thus have

$$\begin{aligned}hnC_{hn}(d) &= \max_{p_{X_1^{hn}}} I(X_1^{hn}; Y) \\ &= \max_{p_{X_1^{hn}}} [I(X_1^{hn}; \tilde{Y}_{(0)}^{(h-1)}) - I(X_1^{hn}; \tilde{Y}_{(0)}^{(h-1)}|Y)] \\ &\geq \max_{p_{X_1^{hn}}} [I(X_1^{hn}; \tilde{Y}_{(0)}^{(h-1)}) - H(\tilde{Y}_{(0)}^{(h-1)}|Y)] \\ &= \max_{p_{X_1^{hn}}} [I(X_1^{hn}; \tilde{Y}_{(0)}^{(h-1)}) - H(L_0^{h-1}|Y)] \\ &\geq \max_{p_{X_1^{hn}}} I(X_1^{hn}; \tilde{Y}_{(0)}^{(h-1)}) - (h-1)\log(n+1) \\ &= hnC_n(d) - (h-1)\log(n+1).\end{aligned}$$

Hence

$$\begin{aligned}C(d) &= \lim_{h\to\infty} C_{hn}(d) \\ &\geq \lim_{h\to\infty} \left[C_n(d) - \frac{h-1}{h} \frac{\log(n+1)}{n} \right] \\ &= C_n(d) - \frac{\log(n+1)}{n}.\end{aligned}$$

■

See [6, eq. (39)] for tighter, though more complicated, bound.

As a consequence of Lemma 1 we have the following regularity result for $C(d)$.

Lemma 2: The function $C(d)$ is uniformly continuous in $[0, 1]$. Thus, for every $\beta > 0$ there is a $\alpha = \alpha(\beta)$ such that $|d_1 - d_2| < \alpha \Rightarrow |C(d_1) - C(d_2)| < \beta$.

Proof: As shown in Lemma 1, the functions $C_n(d)$ tend to $C(d)$ uniformly in d. Hence, if proved that the $C_n(d)$ are continuous in d, so is their limit $C(d)$. Since the domain of $C(d)$ is compact, by the Heine-Cantor theorem $C(d)$ is also uniformly continuous. That the $C_n(d)$ functions are continuous is really intuitive; the shortest formal proof that we were able to provide goes as follows. The entries of the transition matrix of the channel W_n^d are polynomials in d and thus the mutual information $I(X_1^n; W_n^d(X_1^n))$ is a continuous function of d and of the input distribution $p_{X_1^n}$. Hence, by moving d continuously from 0 to 1 one expects the capacity to change continuously from 1 to 0. A formal proof, however, seems to require using the compactness of the sets of distributions $p_{X_1^n}$. Assume that $C_n(d)$ is not continuous in $d = \bar{d}$ and let $\bar{p}$ be the input distribution that attains the value $C_n(\bar{d})$. Then there exists an $\varepsilon > 0$ such that $|C_n(\bar{d}) - C_n(d_k)| > \varepsilon$ for a sequence d_k converging to $\bar{d}$. Consider the distributions p_k that attain $C_n(d_k)$. Since the set of the $p_{X_1^n}$ is bounded and closed, there exists a subsequence of the p_k that converges to a distribution p'. By continuity of the mutual information the $C_n(d_k)$ values tend to the mutual information I' attained by p' in $d = \bar{d}$. But, by definition of $C_n(\bar{d})$, we clearly have that $I' \leq C_n(\bar{d})$ and thus $C_n(d_k) \leq C_n(\bar{d}) - \varepsilon$ for k large enough. But then the mutual information attained by $\bar{p}$ in d_k tends to $C_n(\bar{d}) \geq C_n(d_k) + \varepsilon$ for large enough k, which is absurd by definition of $C_n(d_k)$. ■

III. EXACT DELETION CHANNEL

Let now $W_{n,k}$, $k \leq n$, be a channel with n-bits input whose output is uniformly chosen within the $\binom{n}{k}$ k-bits subsequences of the input. This channel was efficiently used as an auxiliary channel in [5], [6]. Let then

$$C_{n,k} = \frac{1}{n} \max_{p_{X_1^n}} I(X_1^n; W_{n,k}(X_1^n)). \tag{7}$$

The following obvious result will be used later.

Lemma 3: For every random X_1^n, if $k_1 \geq k_2$ then

$$I(X_1^n; W_{n,k_1}(X_1^n)) \geq I(X_1^n; W_{n,k_2}(X_1^n)). \tag{8}$$

Proof: Simply note that the W_{n,k_2} channel can be obtained as a cascade of W_{n,k_1} and W_{k_1,k_2}. Thus, $X_1^n \to W_{n,k_1}(X_1^n) \to W_{n,k_2}(X_1^n)$ is a Markov chain and the lemma follows from the data processing inequality. ■

The following lemma bounds the capacity of the W_n^d channel in terms of the capacity of certain exact deletion channels.

Lemma 4: For every $\varepsilon > 0$, $d \in [\varepsilon, 1-\varepsilon]$, and $n \geq 1$

$$C_{n,\lceil(1-d-\varepsilon)n\rceil} - 2e^{-2\varepsilon^2 n} \leq C_n(d) \leq C_{n,\lfloor(1-d+\varepsilon)n\rfloor} + 2e^{-2\varepsilon^2 n}. \tag{9}$$

Proof: We first prove the right hand side inequality. For an input X_1^n, let $Y = W_n^d(X_1^n)$ and let $L = |Y|$ be the length of Y. First note that $X_1^n \to Y \to L$ is a Markov chain. So, by applying the chain rule to $I(X_1^n;Y,L)$, considered that $I(X_1^n;L) = 0$ since L is independent from X_1^n, it is easily seen that $I(X_1^n;Y) = I(X_1^n;Y|L)$. Define $T = \{j : \left|\frac{j}{n} - (1-d)\right| \leq \varepsilon\}$, that is $j \in T$ if and only if $\lceil(1-d-\varepsilon)n\rceil \leq j \leq \lfloor(1-d+\varepsilon)n\rfloor$. Let now X_1^n be distributed according to the optimal distribution for the W_n^d channel. Then we have

$$\begin{aligned}
nC_n(d) &= I(X_1^n;Y|L) \\
&= \sum_{j=0}^{n} p_L(j) I(X_1^n;Y|L=j) \\
&= \sum_{j\in T} p_L(j) I(X_1^n;Y|L=j) \\
&\qquad + \sum_{j\in \bar{T}} p_L(j) I(X_1^n;Y|L=j) \\
&\stackrel{(a)}{\leq} \sum_{j\in T} p_L(j) I(X_1^n;Y|L=\lfloor(1-d+\varepsilon)n\rfloor) \\
&\qquad + \sum_{j\in \bar{T}} p_L(j) n \\
&\leq nC_{n,\lfloor(1-d+\varepsilon)n\rfloor} \sum_{j\in T} p_L(j) + n \sum_{j\in \bar{T}} p_L(j) \\
&\stackrel{(b)}{\leq} nC_{n,\lfloor(1-d+\varepsilon)n\rfloor} + 2ne^{-2\varepsilon^2 n},
\end{aligned}$$

where (a) follows from Lemma 3 and the definition of T and (b) follows from the Chernoff bound. Dividing by n we get the desired inequality.

As for the left hand side inequality, let now X_1^n be distributed according to the optimal distribution for the $W_{n,\lceil(1-d-\varepsilon)n\rceil}$ channel. Then we have

$$\begin{aligned}
nC_n(d) &\geq I(X_1^n;Y|L) \\
&= \sum_{j=0}^{n} p_L(j) I(X_1^n;Y|L=j) \\
&= \sum_{j\in T} p_L(j) I(X_1^n;Y|L=j) \\
&\qquad + \sum_{j\in \bar{T}} p_L(j) I(X_1^n;Y|L=j) \\
&\stackrel{(a)}{\geq} \sum_{j\in T} p_L(j) I(X_1^n;Y|L=\lceil(1-d-\varepsilon)n\rceil) \\
&= nC_{n,\lceil(1-d-\varepsilon)n\rceil} \sum_{j\in T} p_L(j) \\
&\stackrel{(b)}{\geq} nC_{n,\lceil(1-d+\varepsilon)n\rceil}(1-2e^{-2\varepsilon^2 n}) \\
&\stackrel{(c)}{\geq} nC_{n,\lceil(1-d+\varepsilon)n\rceil} - 2ne^{-2\varepsilon^2 n},
\end{aligned}$$

where (a) follows again from Lemma 3, (b) follows from the Chernoff bound, and (c) follows from the obvious fact that $C_{n,\lceil(1-d+\varepsilon)n\rceil} \leq 1$. Dividing by n the desired result is obtained. ∎

The following lemma bounds the capacity of the exact deletion channel $W_{n,k}$ in terms of $C(d)$ for appropriate values of d.

Lemma 5: For every $\varepsilon > 0$ and integers n and k

$$\begin{aligned}
C\left(1-k/n+\varepsilon\right) - 2e^{-2\varepsilon^2 n} \leq C_{n,k} &\leq C\left(1-k/n-\varepsilon\right) \\
&\quad + 2e^{-2\varepsilon^2 n} + \frac{\log(n+1)}{n}. \quad (10)
\end{aligned}$$

Proof: Take $d = 1-k/n-\varepsilon$ in Lemma 4 to obtain $C_{n,k} \leq C_n(1-k/n-\varepsilon) + 2e^{-2\varepsilon^2 n} \leq C(1-k/n-\varepsilon) + 2e^{-2\varepsilon^2 n} + \log(n+1)/n$, by virtue of Lemma 1. Then take $d = 1-k/n+\varepsilon$ in Lemma 4 to obtain $C_{n,k} \geq C_n(1-k/n+\varepsilon) - 2e^{-2\varepsilon^2 n} \geq C(1-k/n+\varepsilon) - 2e^{-2\varepsilon^2 n}$. ∎

Lemma 6: For every $\beta > 0$, there is an $\bar{n} = \bar{n}(\beta)$ such that

$$|C_{n,k} - C(1-k/n)| < \beta \qquad \forall n \geq \bar{n},\ k = 1,\dots,n. \quad (11)$$

Proof: First note that, for $\varepsilon > 0$, $C(1-k/n+\varepsilon) \leq C(1-k/n) \leq C(1-k/n-\varepsilon)$. Hence, $C(1-k/n)$ satisfies the two inequalities satisfied by $C_{n,k}$ in equation (10). So, $|C_{n,k} - C(1-k/n)|$ is bounded by the difference between the right hand side and the left hand side of equation (10), that is

$$\begin{aligned}
|C_{n,k} - C(1-k/n)| &\leq C\left(1-k/n-\varepsilon\right) - C\left(1-k/n+\varepsilon\right) \\
&\quad + 4e^{-2\varepsilon^2 n} + \frac{\log(n+1)}{n}. \quad (12)
\end{aligned}$$

With the notation of Lemma 2, take $\varepsilon < \alpha(\beta/2)/2$ so that $C\left(1-k/n-\varepsilon\right) - C\left(1-k/n+\varepsilon\right) < \beta/2$. Once ε is fixed, choose $\bar{n}$ such that $4e^{-2\varepsilon^2 \bar{n}} + \frac{\log(\bar{n}+1)}{\bar{n}} < \beta/2$ to complete the proof. Note that $\bar{n}$ is a function of β only and that the result holds for every $k \leq n$. ∎

We can now state the first result of this paper.

Theorem 1: Let k_n be an integer valued sequence such that k_n/n tends to $1-d$ as n goes to infinity. Then

$$\lim_{n\to\infty} C_{n,k_n} = C(d). \quad (13)$$

Proof: It follows easily from Lemma 6 by continuity of $C(d)$. ∎

IV. Behavior near $d = 1$

In this Section, we finally focus on the behavior of the function $C(d)$ for values of d close to 1. It is interesting to observe in Figure 1 that, from experimental evidence, the $C_n(d)$ functions seem to be convex in a progressively expanding region of d values. On the one hand, it is tempting to conjecture that the limit $C(d)$ is convex in the whole interval $d \in [0,1]$. On the other hand, near $d=1$, all the $C_n(d)$ curves appear to change concavity and go to zero asymptotically as $(1-d)$. Indeed, we have the following result.

Lemma 7: For every n,

$$\lim_{d\to 1} \frac{C_n(d)}{(1-d)} = 1 \quad (14)$$

Proof: It is easily shown that for every n and d

$$(1 - d^n)/n \leq C_n(d) < (1 - d). \tag{15}$$

The right hand side inequality follows from the fact that the capacity of W_n^d is obviously smaller than the capacity of a binary erasure channel with erasure probability d. To prove the left hand side inequality consider using as input to the channel W_n^d only the sequence composed of n zeros and that composed of n ones. Then the n uses of W_n^d correspond to one use of an erasure channel with erasure probability d^n. This proves equation (15). Dividing by $(1-d)$ and taking the limit $d \to 1$ gives the required result. ■

Lemma 7 ensures that, for fixed n, $C_n(d)$ is not convex in a neighborhood of $d = 1$. Note further that

$$\lim_{d \to 1} \frac{C_n(d)}{(1-d)} = \sup_{d \in (0,1)} \frac{C_n(d)}{(1-d)} = 1 \tag{16}$$

Hence, it is natural to believe that $C_n(d)$ is actually concave in a neighborhood of $d = 1$, even if Lemma 7 is not sufficient to prove this. However, in the limit $n \to \infty$, it is known (see [7], [6]) that $C(d)$ satisfies

$$0.1185 \leq \liminf_{d \to 1} \frac{C(d)}{1-d} \leq \limsup_{d \to 1} \frac{C(d)}{1-d} \leq 0.49 \tag{17}$$

Hence, Lemma 7 does not hold with $C(d)$ in place of $C_n(d)$ and it is still legitimate to conjecture that $C(d)$ may be convex in $[0, 1]$. The next step is thus to ask if $C_n(d)/(1-d)$ has a limit as $d \to 1$ and, if so, if this limit is reached from above as would be implied by convexity of $C(d)$. The remaining part of this section tries to answer this question.

In order to understand the behavior of $C(d)$ near $d = 1$, the following result from [6] is fundamental.

Lemma 8 (Fertonani and Duman, [6, eq. (32)]): For every n, k

$$\limsup_{d \to 1} \frac{C(d)}{1-d} \leq \frac{nC_{n,k} + 1}{k+1}. \tag{18}$$

Remark 1: In [6] the authors state that, for every n and k, $\lim_{d \to 1} \frac{C(d)}{1-d} \leq \frac{nC_{n,k}+1}{k+1}$. However, we are not aware of a previous formal proof that $\lim_{d \to 1} \frac{C(d)}{1-d}$ exists. This fact is proved in the following theorem.

Theorem 2: It holds that

$$\lim_{d \to 1} \frac{C(d)}{(1-d)} = \inf_{d \in (0,1)} \frac{C(d)}{1-d}. \tag{19}$$

Proof: For every $d' \in (0, 1)$, let k_n be a sequence such that k_n/n tends to $1 - d'$. Then, from Theorem 1, the right hand side of (18), with k_n in place of k, tends to $C(d')/(1 - d')$. Since d' is arbitrary, Lemma 8 implies that $\limsup_{d \to 1} C(d)/(1 - d) \leq \inf_{d' \in (0,1)} \frac{C(d')}{1-d'}$. However, it is obvious that $\liminf_{d \to 1} C(d)/(1-d) \geq \inf_{d' \in (0,1)} \frac{C(d')}{1-d'}$. Thus $\lim_{d \to 1} C(d)/(1 - d)$ exists and is equal to $\inf_{d' \in (0,1)} \frac{C(d')}{1-d'}$ ■

A direct consequence of Theorem 2 is the following improved bound on $C(d)$.

Corollary 1:

$$\lim_{d \to 1} \frac{C(d)}{(1-d)} \leq 0.4143. \tag{20}$$

Proof: As far as the author knows, the best known numerical bound obtained for $\inf_d C(d)/(1 - d)$ is 0.4143 obtained using the bound $C(0.65) \leq C_{17}(0.65) = 0.145$, numerically evaluated in [6]. ■

The usefulness of Theorem 2 is that it allows to deduce provable bounds for $\lim_{d \to 1} \frac{C(d)}{(1-d)}$ from bounds on $C(d)$ even with d much smaller than 1. It is interesting to note, in fact, that different techniques seem to be effective in bounding $C(d)$ in different regions of the interval $[0, 1]$. For example, different genie aided channels are used in [6] for smaller values of d than for large values of d and, while equation (18) is derived in [6] using a bound effective for large d, the bound for $C(0.65)$ used in Corollary 1 is derived from the numerical value of $C_{17}(d)$ which is not as effective for d larger than 0.8 (see Table IV in [6], where bound C_4 therein is what we called $C_{17}(d)$, while bound C_2^* is used to deduce (18)). Thus, in order to obtain improved upper bounds for $\lim_{d \to 1} \frac{C(d)}{(1-d)}$ one effective approach would be to numerically evaluate $C_n(d)$ near $d = 0.65$ for $n \geq 18$. This requires, however, high computational and spatial complexity and it is out of the scope of the present paper.

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